Computation of Nash Equilibria: Two-Player Games

Argyrios Deligkas

Royal Holloway, University of London

argyrios.deligkas@rhul.ac.uk

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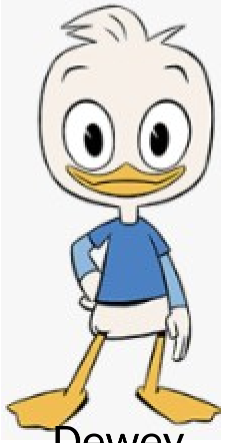
Outline

- Two-player games: The Basics
- Two-player games: Algorithms and Complexity Issues
- Many-player games
  - Normal form games
  - Polymatrix games

Two-player games: The Basics

Dewey and Huey face a dilemma

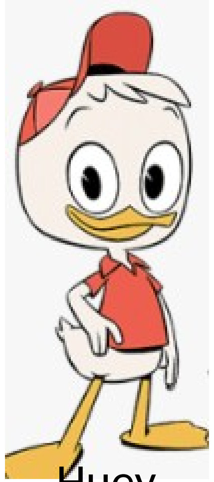
Uncle Scrooge found out that a penny was missing from his vault.
Dewey and Huey were accused of taking it!
Each one can either *admit* or *deny* he took it



Dewey

- If Dewey admits and Huey admits, then Dewey will be suspended for 2 hours
- If Dewey admits and Huey denies, then Dewey will be suspended for 0 hours
- If Dewey denies and Huey admits, then Dewey will be suspended for 3 hours
- If Dewey denies and Huey denies, then Dewey will be suspended for 1 hour

	denies	admits
denies	-1	-3
admits	0	-2



Huey

- If Dewey admits and Huey admits, then Huey will be suspended for 2 hours
- If Dewey admits and Huey denies, then Huey will be suspended for 3 hours
- If Dewey denies and Huey admits, then Huey will be suspended for 0 hours
- If Dewey denies and Huey denies, then Huey will be suspended for 1 hour

	denies	admits
denies	-1	0
admits	-3	-2

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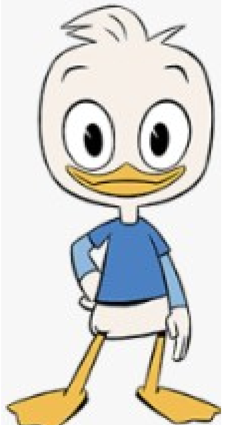

Dewey

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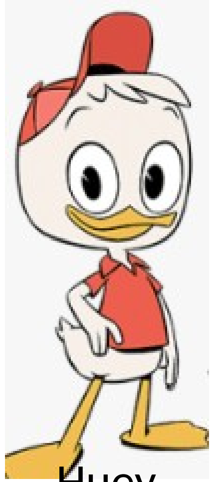

Huey

	denies	admits
denies	-1	0
admits	-3	-2

	denies	admits
denies	-1, -1	0, -3
admits	-3, 0	-2, -2

- Each one of them want to minimize their individual suspension time!
- Each one is clever!
- Uncle Scrooge keeps them in separate rooms so they cannot communicate!

What should they choose????

Dewey and Huey face a dilemma

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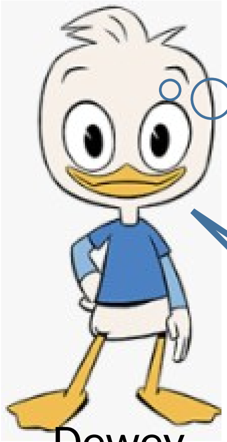
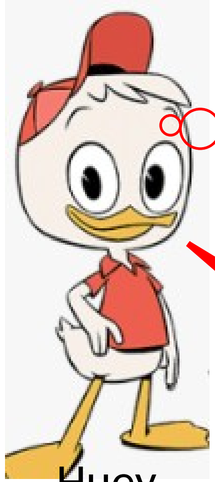


Dewey

- If I deny, then Huey should admit. Thus, I'll get suspended for 3 hours!
- If I admit, then Huey should admit. Thus I'll get suspended for 2 hours!

So I will admit!



Huey

- If I deny, then Dewey should admit. Thus, I'll get suspended for 3 hours!
- If I admit, then Dewey should admit. Thus I'll get suspended for 2 hours!

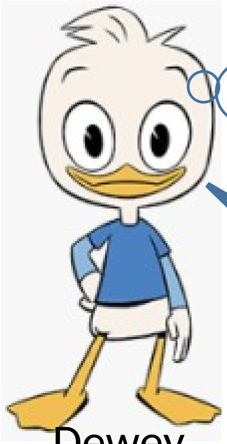
So I will admit!

	denies	admits
denies	-1; -1	0; -3
admits	-3; 0	-2; -2

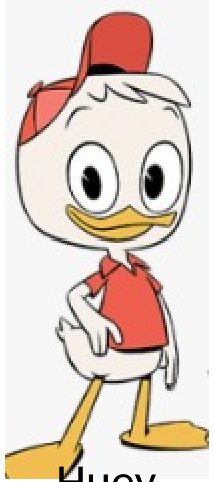
- Each one of them want to minimize their individual suspension time!
- Each one is clever!
- Uncle Scrooge keeps them in separate rooms so they cannot communicate!

What should they choose????

Dewey and Huey play Rock-Paper-Scissors



Dewey



Huey

- If I play Rock, then Huey will play Paper,
so then I will have to play Scissors,
but then Huey will play Rock,
so then I will have to play Paper,
but then Huey will play Scissors,
so then I will have to play Rock,
BUT THEN...

WHAT SHOULD I PLAY??
Shall I play at random???

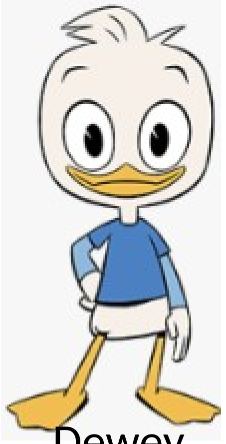
	Rock	Paper	Scissors
Rock	0 0	1 -1	-1 1
Paper	-1 1	0 0	1 -1
Scissors	1 -1	-1 1	0 0

- Each one wants to maximize his score!

What should they choose????

Two-Player Games (Bimatrix Games)

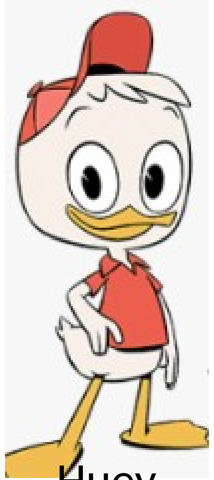
- Two players: Row player and Column player



Dewey

- A set of actions for every player
  Row player has actions
  Column player has actions

- Payoff matrix for every player
  for the Row player of size
   for the Column player of size



Huey

- is the payoff the Row player gets when
  Row player chooses action and
  Column player chooses action

- is the payoff the Column player gets when
  Row player chooses action and
  Column player chooses action

	L	R
T	3, 3	3, 2
M	2, 2	5, 6
B	0, 3	6, 1

Row player has 3 actions: T, M, B
Column player has 2 actions: L, R

Two-Player Games - Strategies

- Two players: Row player and Column player
- A set of actions for every player
  Row player has actions
  Column player has actions
- Payoff matrix for every player
  for the Row player of size
  for the Column player of size
- is the payoff the Row player gets when
  Row player chooses action and
  Column player chooses action
- is the payoff the Column player gets when
  Row player chooses action and
  Column player chooses action

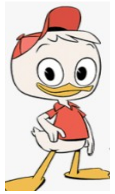
To play the game:
- Row player chooses action
- Column player chooses action

They can choose an action *probabilistically*!

- Row player chooses his action according
  to probability distribution ;
  is the probability he chooses action

- Column player chooses his action according
  to probability distribution ;
  is the probability he chooses action

is the strategy of Row player
is the strategy of Column player
is the strategy profile
is a *pure* strategy if for some



	y_1 L	y_2 R
x_1 T	3 3	2 3
x_2 M	2 2	6 5
x_3 B	3 0	1 6

Two-Player Games – Expected Payoffs

To play the game:
- Row player chooses action
- Column player chooses action

They can choose an action *probabilistically*!
- Row player chooses his action according to probability distribution ;
  is the probability he chooses action
- Column player chooses his action according to probability distribution ;
  is the probability he chooses action

is the strategy of Row player
is the strategy of Column player
is the strategy profile
is a *pure* strategy if for some

In strategy profile
- The expected payoff of Row player is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i \cdot y_j \cdot R_{ij} = x^T R y$$

- The expected payoff of Column player is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i \cdot y_j \cdot C_{ij} = x^T C y$$

	y_1 L	y_2 R
x_1 T	3, 3	3, 2
x_2 M	2, 2	5, 6
x_3 B	0, 3	6, 1

Two-Player Games - Payoffs

In strategy profile

- The expected payoff of Row player is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i \cdot y_j \cdot R_{ij} = x^T Ry$$

$$x^T Ry = \sum_{i=1}^{m} x_i \sum_{j=1}^{n} R_{ij} \cdot y_j = ¿ \sum_{i=1}^{m} x_i \cdot \left(Ry \right)_i ¿$$

Expected payoff of action

- The expected payoff of Column player is

$$\sum_{i=1}^{m}\sum_{j=1}^{n} x_i \cdot y_j \cdot C_{ij} = x^T Cy$$

$$x^T C y = \sum_{j=1}^{n} y_j \sum_{i=1}^{m} C_{ij} \cdot x_i = ¿ \sum_{j=1}^{n} y_j \cdot \left(C^T x \right)_j ¿$$

Expected payoff of action

	y_1 L	y_2 R	
x_1 T	3 (row), 3 (column)	3 (row), 2 (column)	$\left(Ry\right)_1 = 3y_1 + 3y_2$
x_2 M	2 (row), 2 (column)	5 (row), 6 (column)	$\left(Ry\right)_2 = 2y_1 + 5y_2$
x_3 B	0 (row), 3 (column)	6 (row), 1 (column)	$\left(Ry\right)_3 = 0y_1 + 6y_2$

$$\left(C^T x\right)_1 = 3x_1 + 2x_2 + 3x_3$$

$$\left(C^T x\right)_2 = 2x_1 + 6x_2 + 1x_3$$

Two-Player Games – Best Responses/Supports

In strategy profile

- The expected payoff of Row player is

$$x^T R y = \sum_{i=1}^{m} x_i \cdot (Ry)_i$$

Expected payoff of action

- The expected payoff of Column player is

$$x^T C y = \sum_{j=1}^{n} y_j \cdot (C^T x)_j$$

Expected payoff of action

Best responses

- Given a strategy for Column player action is a *pure best response* if
- Given a strategy for Row player action is a *pure best response* if

Regret

- The *regret* of Row player under is
- The *regret* of Column player under is

Support

- The *support* of a strategy is the set of actions played with positive probability
- The *support* of a strategy is the set of actions played with positive probability

Two-Player Games – Best Responses/Supports

Best responses
- Given a strategy for Column player action is a *pure best response* if
- Given a strategy for Row player action is a *pure best response* if

Support
- The *support* of a strategy is the set of actions played with positive probability
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Regret
- The *regret* of Row player under is
- The *regret* of Column player under is

$$supp(x)=\{T,M\}$$

		2/3 (L)	1/3 (R)
1/2	T	3, 3	3, 2
1/2	M	2, 2	5, 6
0	B	0, 3	6, 1

$$x^T C y = \frac{2}{3}\frac{5}{2} + \frac{1}{3}\frac{8}{2} = \frac{18}{6}$$

Regret for Col:

$$(R \cdot y)_1 = 3 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = \frac{9}{3}$$

$$(R \cdot y)_2 = 2 \cdot \frac{2}{3} + 5 \cdot \frac{1}{3} = \frac{9}{3}$$

$$(R \cdot y)_3 = 0 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{6}{3}$$

$$x^T R y = \frac{1}{2}\frac{9}{3} + \frac{1}{2}\frac{9}{3} + 0 \cdot \frac{6}{3} = \frac{9}{3}$$

Regret for Row:

$$(x \cdot C^T)_1 = 3 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + 3 \cdot 0 = \frac{5}{2}$$

$$(x \cdot C^T)_2 = 2 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} + 1 \cdot 0 = \frac{8}{2}$$

Two-Player Games – Nash Equilibrium

Best responses
- Given a strategy for Column player action is a *pure best response* if
- Given a strategy for Row player action is a *pure best response* if

Support
- The *support* of a strategy is the set of actions played with positive probability
- The *support* of a strategy is the set of actions played with positive probability

Regret
- The *regret* of Row player under is
- The *regret* of Column player under is

is a Nash equilibrium of a bimatrix game if one of the following holds (equivalent definitions)

Both players play a (mixed) best response

is a best response against

is a best response against

The supports of both players contain only pure best responses

$$\hat{i} \in supp(x) \Rightarrow (Ry)_{\hat{i}} = max_i (Ry)_i$$

$$\hat{j} \in supp(y) \Rightarrow (C^T x)_{\hat{j}} = max_j (C^T x)_j$$

The regret of every player is zero

$$max_i (Ry)_i - x^T Ry = 0$$

= 0

At equilibrium no player can improve their payoff by unilaterally changing their strategy

Two-Player Games – Nash Equilibrium

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if one of the following holds (equivalent definitions)

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= 0

Theorem (Nash)

Every *finite* game possesses at least one Nash equilibrium
- finite number of players
- finite number of actions for every player

Two-Player Games – Nash Equilibrium

Theorem (Nash)

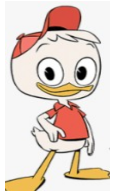
Every *finite* game possesses at least one Nash equilibrium
- finite number of players
- ~~finite number of~~ actions for every player

Consider the two-player game:
- the actions of each player is any number in (0,1)
- the payoff of a player is
  1 if they chose the lower number of the two
  0 if they chose the higher number of the two

Is there a Nash equilibrium in the game above?

NO!

Two-Player Games – Example



		$2/3$	$1/3$
		L	R
$1/2$	T	3 (Row), 3 (Col)	3 (Row), 2 (Col)
$1/2$	M	2 (Row), 2 (Col)	5 (Row), 6 (Col)
0	B	0 (Row), 3 (Col)	6 (Row), 1 (Col)

$$x^T C y = \frac{2}{3}\frac{5}{2} + \frac{1}{3}\frac{8}{2} = \frac{18}{6}$$

Regret for Col:

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$$x^T R y = \frac{1}{2}\frac{9}{3} + \frac{1}{2}\frac{9}{3} + 0 \cdot \frac{6}{3} = \frac{9}{3}$$

Regret for Row:

$$(x^T \cdot C)_1 = 3 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} + 3 \cdot 0 = \frac{5}{2}$$

$$(x^T \cdot C)_2 = 2 \cdot \frac{1}{2} + 6 \cdot \frac{1}{2} + 1 \cdot 0 = \frac{8}{2}$$

Two-Player Games – Example

The supports of both players contain only pure best responses

$$\hat{i} \in supp(x) \Rightarrow (Ry)_{\hat{i}} = max_i (Ry)_i$$

$$\hat{j} \in supp(y) \Rightarrow (C^T x)_{\hat{j}} = max_j (C^T x)_j$$

The regret of every player is zero

$$max_i (Ry)_i - x^T Ry = 0$$

= 0

		2/3	1/3
		L	R
45	T	3 (Row), 3 (Col)	3 (Row), 2 (Col)
1/5	M	2 (Row), 2 (Col)	5 (Row), 6 (Col)
0	B	0 (Row), 3 (Col)	6 (Row), 1 (Col)

$$x^T C y = \frac{2}{3}\frac{14}{5} + \frac{1}{3}\frac{14}{5} = \frac{14}{5}$$

Regret for Col:

$$(R \cdot y)_1 = 3 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} = \frac{9}{3}$$

$$(R \cdot y)_2 = 2 \cdot \frac{2}{3} + 5 \cdot \frac{1}{3} = \frac{9}{3}$$

$$(R \cdot y)_3 = 0 \cdot \frac{2}{3} + 6 \cdot \frac{1}{3} = \frac{6}{3}$$

$$x^T Ry = \frac{4}{5}\frac{9}{3} + \frac{1}{5}\frac{9}{3} + 0 \cdot \frac{6}{3} = \frac{9}{3}$$

Regret for Row:

$$(x^T \cdot C)_1 = 3 \cdot \frac{4}{5} + 2 \cdot \frac{1}{5} + 3 \cdot 0 = \frac{14}{5}$$

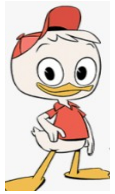
$$(x^T \cdot C)_2 = 2 \cdot \frac{4}{5} + 6 \cdot \frac{1}{5} + 1 \cdot 0 = \frac{14}{5}$$

Two-Player Games – Example

		2/3	1/3
		L	R
45	T	3, 3	3, 2
1/5	M	2, 2	5, 6
0	B	0, 3	6, 1

$x^T R y = 3 \quad x^T C y = \frac{15}{4}$

Best NE for Column



		1/3	2/3
		L	R
0	T	3, 3	3, 2
1/3	M	2, 2	5, 6
2/3	B	0, 3	6, 1

$x^T R y = 4 \quad x^T C y = \frac{8}{3}$

Best NE for Row

		1	0
		L	R
1	T	3, 3	3, 2
0	M	2, 2	5, 6
0	B	0, 3	6, 1

$x^T R y = 3 \quad x^T C y = 3$

Two-Player Games – Approximate NE

is an NE of if

The supports of both players contain only pure best responses

$$\hat{i} \in \boldsymbol{supp}(\boldsymbol{x}) \Rightarrow (\boldsymbol{Ry})_{\hat{i}} = \boldsymbol{max}_{i}(\boldsymbol{Ry})_{i}$$

$$\hat{j} \in \boldsymbol{supp}(\boldsymbol{y}) \Rightarrow (\boldsymbol{C}^{T}\boldsymbol{x})_{\hat{j}} = \boldsymbol{max}_{j}(\boldsymbol{C}^{T}\boldsymbol{x})_{j}$$

is an NE of if

The regret of every player is zero

$$\boldsymbol{max}_{i}(\boldsymbol{Ry})_{i} - \boldsymbol{x}^{T}\boldsymbol{Ry} = \boldsymbol{0}$$

= 0

is an -Well-Supported NE of if

The supports of both players contain only -best responses

$$\hat{i} \in \boldsymbol{supp}(\boldsymbol{x}) \Rightarrow (\boldsymbol{Ry})_{\hat{i}} \geq \boldsymbol{max}_{i}(\boldsymbol{Ry})_{i} - \boldsymbol{\epsilon}$$

$$\hat{j} \in \boldsymbol{supp}(\boldsymbol{y}) \Rightarrow (\boldsymbol{C}^{T}\boldsymbol{x})_{\hat{j}} \geq \boldsymbol{max}_{j}(\boldsymbol{C}^{T}\boldsymbol{x})_{j} - \boldsymbol{\epsilon}$$

is an -NE of if

The regret of every player is at most

$$\boldsymbol{max}_{i}(\boldsymbol{Ry})_{i} - \boldsymbol{x}^{T}\boldsymbol{Ry} \leq \boldsymbol{\epsilon}$$

$$\boldsymbol{max}_{j}(\boldsymbol{C}^{T}\boldsymbol{x})_{j} - \boldsymbol{x}^{T}\boldsymbol{Cy} \leq \boldsymbol{\epsilon}$$

is an -WSNE

The smaller the the better the approximation!

Two-Player Games – Normalization

is an -Well-Supported NE of if

The supports of both players contain only -best responses

$$\hat{i} \in supp(x) \Rightarrow (Ry)_{\hat{i}} \geq max_i (Ry)_i - \epsilon$$

$$\hat{j} \in supp(y) \Rightarrow (C^T x)_{\hat{j}} \geq max_j (C^T x)_j - \epsilon$$

is an -NE of if

The regret of every player is at most

$$max_i (Ry)_i - x^T Ry \leq \epsilon$$

$$max_j (C^T x)_j - x^T Cy \leq \epsilon$$

Normalization DOES NOT change the NE

	L	R
T	0, 8	1, 9
B	1, 0	2, 7

Regret
- Row: 1 – 0 = 1
- Col: 9 – 8 = 1

“Fair”??

Normalize
- max
- min

- max
- min

	L	R
T	0, 8/9	1/2, 1
B	1/2, 0	1, 7/9

Regret
- Row: 1/2 – 0 = 1/2
- Col: 1 – 8/9 = 1/9

Two-Player Games – Classes of Games

Zero-Sum

$$\boldsymbol{R} = -\boldsymbol{C}$$

0 / 0	1 / -1
-1 / 1	0 / 0
1 / -1	-1 / 1

Solvable via LP

Every NE yields the same payoff for each of the players

Symmetric

$$\boldsymbol{R} = \boldsymbol{C}^T$$

0 / 0	1 / -1	-1 / 1
-1 / 1	0 / 0	1 / -1
1 / -1	-1 / 1	0 / 0

Every symmetric two-player game has a *symmetric* NE, i.e. both players play the same strategy [1]

Win-Lose

$$R_{ij} \in \{0,1\}$$

0 / 0	1 / 0
0 / 1	0 / 0
1 / 0	1 / 1

Coordination

$$\boldsymbol{R}_{ij} = \boldsymbol{C}_{ij}$$

-2 / -2	1 / 1
0 / 0	0 / 0
1 / 1	3 / 3

Easy to solve

[1] Non-cooperative games. Nash

Two-Player Games – Symmetric Games

General Game

	3, 3	3, 2
	2, 2	5, 6

Symmetric Game

	0, 0	R, C
	C^T, R^T	0, 0

	0, 0	0, 0	3, 3	3, 2
	0, 0	0, 0	2, 2	5, 6
	3, 3	2, 2	0, 0	0, 0
	2, 3	6, 5	0, 0	0, 0

Every NE of the Symmetric Game corresponds to an NE of the original game

So, finding an NE in a symmetric game is hard as finding an NE in an arbitrary game

Two-player games:
Algorithms and Complexity Issues

Algorithms for NE – support enumeration

(x,y) is an NE

$\hat{\imath} \in supp(x) \Rightarrow (Ry)_{\hat{\imath}} = max_i (Ry)_i$

$\hat{\jmath} \in supp(y) \Rightarrow (C^T x)_{\hat{\jmath}} = max_j (C^T x)_j$

for every in
for every and

for every

For each possible support of Row player and each possible support of Col player check if the linear system above has a feasible solution

for every in
for every and

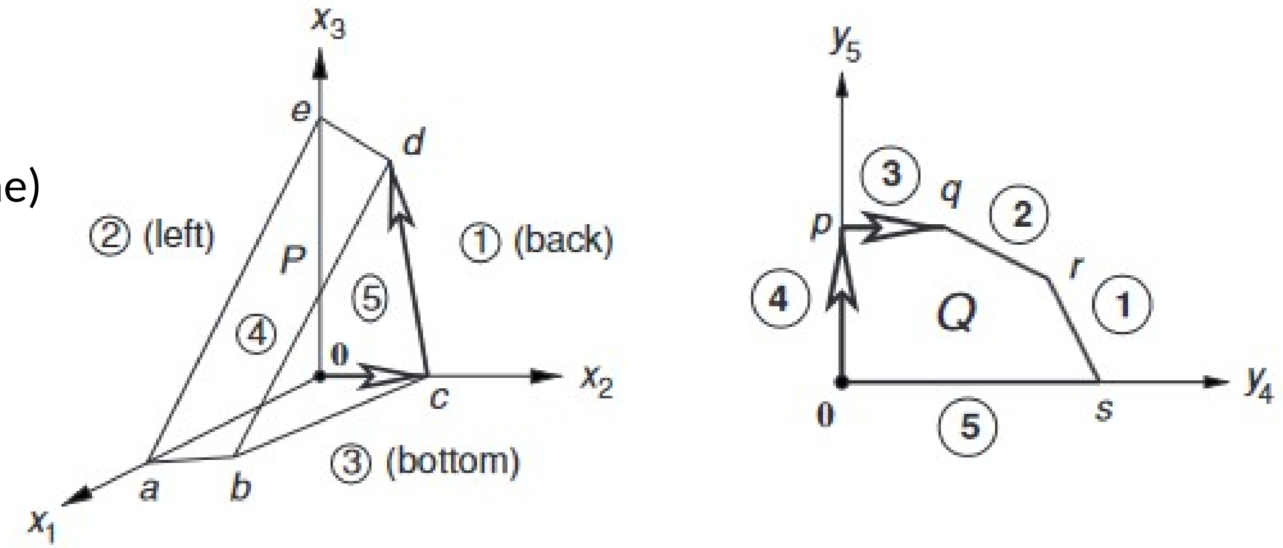
for every

choices!!

Algorithms for NE – Lemke-Howson

- Moves on best-response polyhedral/polytopes
- Performs pivoting on their edges until a NE is reached

(excellent explanation by von Stengel at Chapter 3 of Algorithmic Game Theory book, available freely online)



- “Fast” in practice
- steps in the worst case [2]
- PSPACE-complete to decide whether Lemke-Howson can find a particular NE [3]

Is there an efficient (i.e. polynomial in the size of the game) algorithm for finding an (approximate) NE?

[2] Hard to Solve Bimatrix games. Savani, von Stengel
[3] The Complexity of the Homotopy Method, Equilibrium Selection, and Lemke-Howson Solutions. Goldberg, Papadimitriou, Savani

Complexity of Nash equilibria

Complexity Crash Course

NP-complete
- YES/NO problems **?**
- Verify in polynomial time any solution of the given problem

Polynomial-time algorithm is unlikely for NP-complete problems

$$\max_i (Ry)_i - x^T Ry \leq \epsilon$$
$$\max_j (C^T x)_j - x^T Cy \leq \epsilon$$

NOT a YES/NO problem!

Theorem (Nash)

Every bimatrix game possesses at least one Nash equilibrium

Complexity of constrained Nash equilibria

<u>Complexity Crash Course</u>

NP-complete
- YES/NO problems
- Verify in polynomial time any solution of the given problem

Polynomial-time algorithm is unlikely for NP-complete problems

$$max_i (Ry)_i - x^T Ry = 0$$
$$max_j (C^T x)_j - x^T Cy = 0$$

Problem definition
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ such that $\min(\mathbf{x}^T R\mathbf{y}, \mathbf{x}^T C\mathbf{y}) \geq u$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ with $\text{supp}(\mathbf{x}) \subseteq S$?
Are there two ϵ-NE with TV distance $\geq d$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ with $\max_i \mathbf{x}_i \leq p$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}^T R\mathbf{y} + \mathbf{x}^T C\mathbf{y} \leq v$?
Is there an ϵ-NE $(\mathbf{x}, \mathbf{y})$ such that $\mathbf{x}^T R\mathbf{y} \leq u$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ such that $\lvert\text{supp}(\mathbf{x})\rvert + \lvert\text{supp}(\mathbf{y})\rvert \geq 2k$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ such that $\min\{\lvert\text{supp}(\mathbf{x})\rvert, \lvert\text{supp}(\mathbf{y})\rvert\} \geq k$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ such that $\lvert\text{supp}(\mathbf{x})\rvert \geq k$?
Is there an ϵ-WSNE $(\mathbf{x}, \mathbf{y})$ with $S_R \subseteq \text{supp}(\mathbf{x})$?

It is NP-hard to decide whether a bimatrix game possesses an exact NE that satisfies any of the constraints above even for symmetric win-lose games [4], [5], [6]

[4] Nash and correlated equilibria: Some complexity considerations. Gilboa, Zemel
[5] New complexity results about Nash equilibria. Conitzer, Sandholm
[6] The complexity of Computational Problems about Nash Equilibria in Symmetric Win-Lose Games. Bilo, Mavronicolas

Complexity Classes - TFNP

Complexity Crash Course

NP-complete ?
- YES/NO problems
- Verify in polynomial time any solution of the given problem

Polynomial-time algorithm is unlikely for NP-complete problems

$$\max_i (Ry)_i - x^T R y \leq \epsilon$$
$$\max_j (C^T x)_j - x^T C y \leq \epsilon$$

NOT a YES/NO problem!

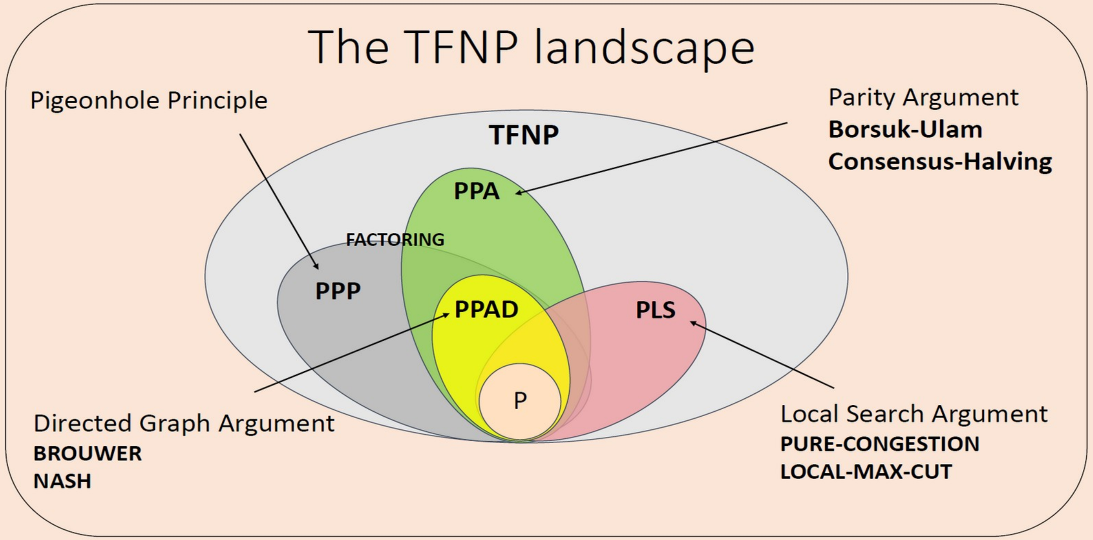
Theorem (Nash)

Every bimatrix game possesses at least one Nash equilibrium

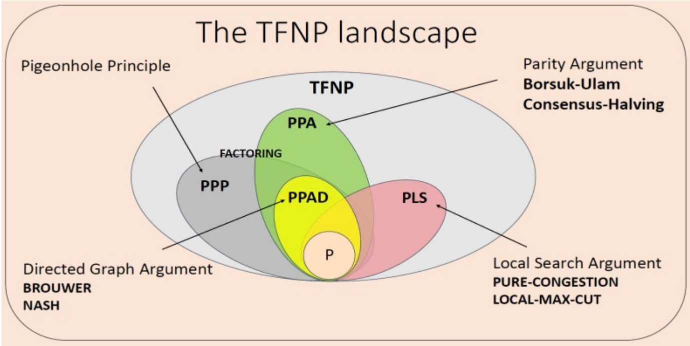
TFNP

Total NP search problems:
- search : looking for a solution, not just YES or NO
- NP: any solution can be checked efficiently
- total: there always exists at least one solution

How do we show that a TFNP-problem is hard:
- No TFNP-problem can be NP-hard, unless NP = coNP…
- Believed that no TFNP-complete problems exists…



Complexity Classes - PPAD



<u>Complexity Crash Course</u>

NP-complete
- YES/NO problems **?**
- Verify in polynomial time any solution of the given problem

Polynomial-time algorithm is unlikely for NP-complete problems

$$max_i (Ry)_i - x^T Ry \leq \epsilon$$
$$max_j (C^T x)_j - x^T Cy \leq \epsilon$$

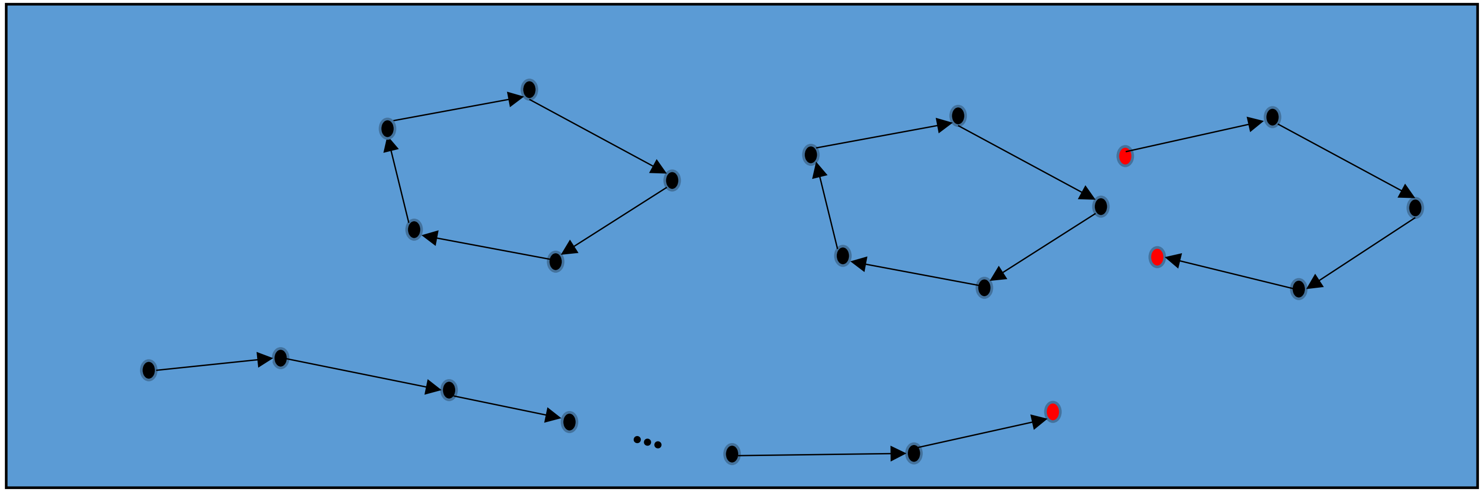
NOT a YES/NO problem!

Theorem (Nash)

Every bimatrix game possesses at least one Nash equilibrium

PPAD (Polynomial Parity Argument Directed) [7]
- YES (i.e. total) problems
- **End-Of-Line** Problem
- Brouwer fixed point

Polynomial-time algorithms are unlikely for PPAD-hard problems



[7] On the Complexity of the Parity Argument and Other Inefficient Proofs of Existence. Papadimitriou

Complexity of Nash equilibria

Polynomial-time algorithms are unlikely for PPAD-hard problems

- **NASH** is PPAD-complete for two-player games [8]
  It is PPAD-hard even for

NASH is PPAD-hard for 4-player games for [9]

- Sparse games
  - NASH is PPAD-complete even for [10]
  Sparse: every row and column of and has at most 10 nonzero entries.

[8] Settling the complexity of computing two-player Nash equilibria. Chen, Deng, Teng
[9] The complexity of computing a Nash equilibrium. Daskalakis, Goldberg, Papadimitriou
[10] Sparse Games are Hard. Chen, Deng, Teng

Complexity of Nash equilibria: Win-Lose

Polynomial-time algorithms are unlikely for PPAD-hard problems

- Win-lose games

  - **NASH** is PPAD-complete [11]

  - **- NASH** for even for [12]

  - poly-time solvable for very sparse games [13]
    at most 2 nonzero entries per row/column

  - poly-time solvable for “planar” games [14]

Win-Lose

$$R_{ij} \in \{0,1\}$$

0 / 0	1 / 0
0 / 1	0 / 0
1 / 0	1 / 1

[11] On the complexity of two-player win-lose games. Abbott, Kane, Valiant
[12] The approximation complexity of win-lose games. Chen, Teng, Valiant
[13] Efficient computation of Nash equilibria for very sparse win-lose bimatrix games. Codenotti, Leoncini, Resta
[14] A polynomial time algorithm for finding Nash equilibria in planar Win-Lose games. Addario-Berry, Olver, Vetta

Complexity of Nash equilibria: rank - k

Polynomial-time algorithms are unlikely for PPAD-hard problems

- Rank - games:

  - Rank - 0: zero-sum. Poly-time solvable

  - FPTAS for constant rank games [15]

  - Rank - 1 games: poly-time solvable [16]

  - Rank 3: PPAD-hard [17]

  - Rank - 2 games? (claimed to be hard, no formal proof known yet)

[15] Games of fixed rank: a hierarchy of bimatrix games. Kannan, Theobald
[16] Fast algorithms for rank-1 bimatrix games. Adsul, Garg, Mehta, Sohoni, von Stengel
[17] Constant rank two-player games are PPAD-hard. Mehta

Complexity of Nash equilibria: Imitation games

Polynomial-time algorithms are unlikely for PPAD-hard problems

- Imitation Game : is the identity matrix [18]

  - PTAS for -WSNE [19]

  - PPAD-hard for for any [19]

1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

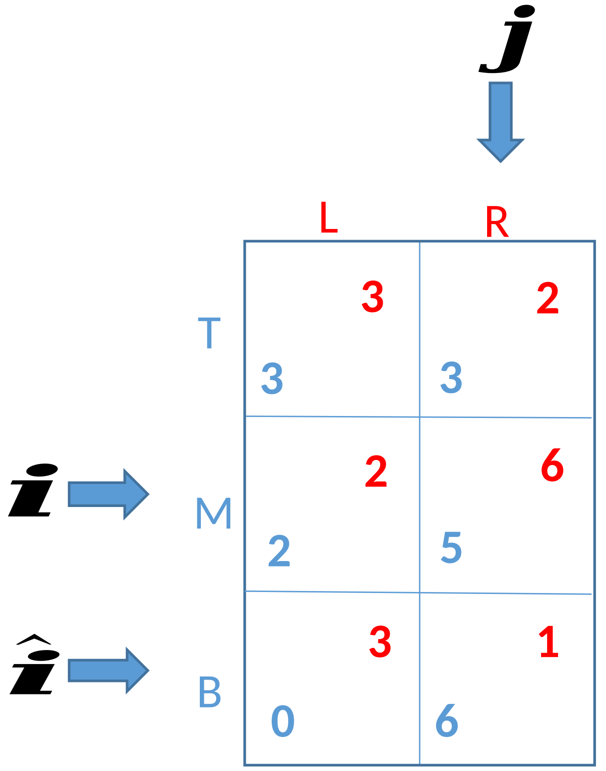
[18] Imitation games and computation. McLennan, Tourky
[19] Approximate Nash Equilibria of Imitation Games: Algorithms and Complexity. Murhekar, Mehta

Algorithms for -NE

- 0.75-NE [20]
- 0.5-NE [21]
- 0.36-NE [22]
- 0.3393-NE [23]
- 1/3-NE [DFM]

DMP algorithm for 0.5-NE

1. Fix a pure strategy for the Row player
2. Compute a best response for the Column player
3. Compute a best response for the Row player
4. Row player plays equiprobably and Column player plays



TS algorithm is based on “gradient descent”. The approximation guarantee is tight [24]

[20] Polynomial Algorithms for Approximating Nash Equilibria of Bimatrix Games. Kontogiannis, Panagopoulou, Spirakis
[21] A Note on Approximate Nash Equilibria. Daskalakis, Mehta, Papadimitriou
[22] New algorithms for approximate Nash equilibria in bimatrix games. Bosse, Byrka, Markakis
[23] An Optimization Approach for Approximate Nash Equilibria. Tsaknakis, Spirakis
[24] On Tightness of the Tsaknakis-Spirakis Algorithm for Approximate Nash Equilibrium. Chen, Deng, Huang, Li, Li
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Algorithms for -WSNE

- 2/3-WSNE [25]
- 0.6608-WSNE [26]
- 0.6528-WSNE [27]
- 0.5-WSNE for **symmetric games** [23]
- 0.5-WSNE [DFM*]

KS algorithm for -WSNE

1. Check if there is a pure profile in that is a 2/3-WSNE

2. If there is not, solve the zero sum game and use the computed strategies

Algorithm 3

1. Solve the zero-sum games $(R, -R)$ and $(-C, C)$.
   - Let $(\mathbf{x}^*, \mathbf{y}^*)$ be a NE of $(R, -R)$, and let $(\hat{\mathbf{x}}, \hat{\mathbf{y}})$ be a NE of $(C, -C)$.
   - Let v_r be the value secured by $\mathbf{x}^*$ in $(R, -R)$, and let v_c be the value secured by $\hat{\mathbf{y}}$ in $(-C, C)$. Without loss of generality assume that $v_c \leq v_r$.
2. If $v_r \leq 2/3 - z$, then return $(\hat{\mathbf{x}}, \mathbf{y}^*)$.
3. If for all $j \in [n]$ it holds that $C_j^T \mathbf{x}^* \leq 2/3 - z$, then return $(\mathbf{x}^*, \mathbf{y}^*)$.
4. Otherwise:
   - Let j^* be a pure best response against $\mathbf{x}^*$. Define:

$$S := \{i \in \mathrm{supp}(\mathbf{x}^*) : R_{i\mathrm{j}*} < 1/3 + z\}$$
$$B := \mathrm{supp}(\mathbf{x}^*) \setminus S$$

   - Define the strategy $\mathbf{x_B}$ as follows. For each $i \in [n]$ we have:

$$(\mathbf{x_B})_i = \begin{cases} \frac{1}{\Pr(B)} \cdot \mathbf{x}_i^* & \text{if } i \in B \\ 0 & \text{otherwise.} \end{cases}$$

   - If $(\mathbf{x_B}^T \cdot C)_{\mathrm{j}*} \geq \frac{1}{3} + z$, then return $(\mathbf{x_B}, \mathrm{j}^*)$.
5. Otherwise:
   - Let j' be a pure best response against $\mathbf{x_B}$.
   - If there exists an $i \in \mathrm{supp}(\mathbf{x}^*)$ such that (i, j^*) or (i, j') is a pure $(\frac{2}{3} - z)$-WSNE, then return it.
   - Find a row $b \in B$ such that $R_{b\mathrm{j}*} > 1 - \frac{18z}{1+3z}$ and $C_{b\mathrm{j}'} > 1 - \frac{18z}{1+3z}$.
   - Find a row $s \in S$ such that $C_{s\mathrm{j}*} > 1 - \frac{27z}{1+3z}$ and $R_{s\mathrm{j}'} > 1 - \frac{27z}{1+3z}$.
   - Define the row player strategy $\mathbf{x_{mp}}$ and the column player strategy $\mathbf{y_{mp}}$ as follows. For each $i \in [n]$ we have:

$$\mathbf{x_{mp}}_i = \begin{cases} \frac{1-24z}{2-39z} & \text{if } i = b, \\ \frac{1-15z}{2-39z} & \text{if } i = s, \\ 0 & \text{otherwise.} \end{cases} \qquad \mathbf{y_{mp}}_i = \begin{cases} \frac{1-24z}{2-39z} & \text{if } i = \mathrm{j}^*, \\ \frac{1-15z}{2-39z} & \text{if } i = \mathrm{j}', \\ 0 & \text{otherwise.} \end{cases}$$

   - Return $(\mathbf{x_{mp}}, \mathbf{y_{mp}})$.

[25] Well Supported Approximate Equilibria in Bimatrix Games. Kontogiannis, Spirakis
[26] Approximate Well-Supported Nash Equilibria Below Two-Thirds. Fearnley, Goldberg, Savani, Sorensen
[27] Distributed Methods for Computing Approximate Equilibria. Czumaj, Deligkas, Fasoulakis, Fearnley, Jurdzinski, Savani
[28] Approximate Well-Supported Nash Equilibria in Symmetric Games. Czumaj, Fasoulakis, Jurdzinski
DFM*: A Polynomial-Time Algoritm for 1/2-Well-Supported Nash Equilibria in Bimatrix Games. Deligkas, Fasoulakis, Markakis

A QPTAS for -NE

Problem description	Problem definition
Large payoffs $u \in (0,1]$	Is there an ϵ-NE (x,y) such that $\min(x^T Ry, x^T Cy) \geq u$?
Small total payoff $v \in [0,2)$	Is there an ϵ-NE (x,y) such that $x^T Ry + x^T Cy \leq v$?
Small payoff $u \in [0,1)$	Is there an ϵ-NE (x,y) such that $x^T Ry \leq u$?
Restricted support $S \subset [n]$	Is there an ϵ-NE (x,y) with $\text{supp}(x) \subseteq S$?
Two ϵ-NE $d \in (0,1]$ apart in Total Variation (TV) distance	Are there two ϵ-NE with TV distance $\geq d$?
Small largest probability $p \in (0,1)$	Is there an ϵ-NE (x,y) with $\max_i x_i \leq p$?
Large total support size $k \in [n]$	Is there an ϵ-WSNE (x,y) such that $\|\text{supp}(x)\| + \|\text{supp}(y)\| \geq 2k$?
Large smallest support size $k \in [n]$	Is there an ϵ-WSNE (x,y) such that $\min\{\|\text{supp}(x)\|, \|\text{supp}(y)\|\} \geq k$?
Large support size $k \in [n]$	Is there an ϵ-WSNE (x,y) such that $\|\text{supp}(x)\| \geq k$?
Restricted support $S_R \subseteq [n]$	Is there an ϵ-WSNE (x,y) with $S_R \subseteq \text{supp}(x)$?

We can find an ϵ-NE in $n^{O\left(\frac{\log n}{\epsilon^2}\right)}$ time [29]

There always exists an ϵ-NE with support size $\log n/\epsilon^2$

- Take any pair of strategies (x, y)
- Randomly sample $\log n/\epsilon^2$ pure strategies
- Play the sampled strategies uniformly
- The resulting payoffs will be within ϵ of the originals w.h.p.

These problems are NP-hard for exact NE!

This technique also gives a QPTAS for constrained NE problems...

[29] Playing large games using simple strategies. Lipton, Markakis, Mehta

A QP Lower Bounds for constrained -NE

Let BestSW(ϵ) be the best social welfare achievable by an ϵ-NE

The problem ϵ-**NE** δ-**SW**

- Find an ϵ-NE
- Whose social welfare is at least BestSW(ϵ) $- \delta$

We have an $n^{O\left(\frac{\log n}{\epsilon^2}\right)}$ time algorithm for ϵ-NE ϵ-SW

If ETH is true then ϵ-NE ϵ-SW requires $n^{\text{poly}(\epsilon)\cdot(\log n)^{1-o(1)}}$ time [30]

Exponential-time hypothesis: 3SAT requires $2^{O(n)}$ time

- Implies every NP-complete problem requires $2^{O(\sqrt[c]{n})}$ time
- Stronger conjecture than P $\neq$ NP

[30] Approximating the best Nash equilibrium in -time breaks the exponential time hypothesis. Braverman, Ko, Weinstein

A QP Lower Bounds for constrained -NE

If ETH is true

all of these problems require $n^{O(\log n)}$ time when $\epsilon < \frac{1}{8}$ [31]

Problem description	Problem definition
Large payoffs $u \in (0,1]$	Is there an ϵ-NE (x,y) such that $\min(x^T Ry, x^T Cy) \geq u$?
Small total payoff $v \in [0,2)$	Is there an ϵ-NE (x,y) such that $x^T Ry + x^T Cy \leq v$?
Small payoff $u \in [0,1)$	Is there an ϵ-NE (x,y) such that $x^T Ry \leq u$?
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Large total support size $k \in [n]$	Is there an ϵ-WSNE (x,y) such that $\lvert\text{supp}(x)\rvert + \lvert\text{supp}(y)\rvert \geq 2k$?
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Exponential-time hypothesis: 3SAT requires $2^{O(n)}$ time

- Implies every NP-complete problem requires $2^{O(\sqrt[c]{n})}$ time
- Stronger conjecture than $P \neq NP$

[31] Inapproximability results for constrained approximate Nash equilibria.

Deligkas, Fearnley, Savani

A QPTAS for -NE

We can find an ϵ-NE in $n^{O\left(\frac{\log n}{\epsilon^2}\right)}$ time

There always exists an ϵ-NE with support size $\log n/\epsilon^2$

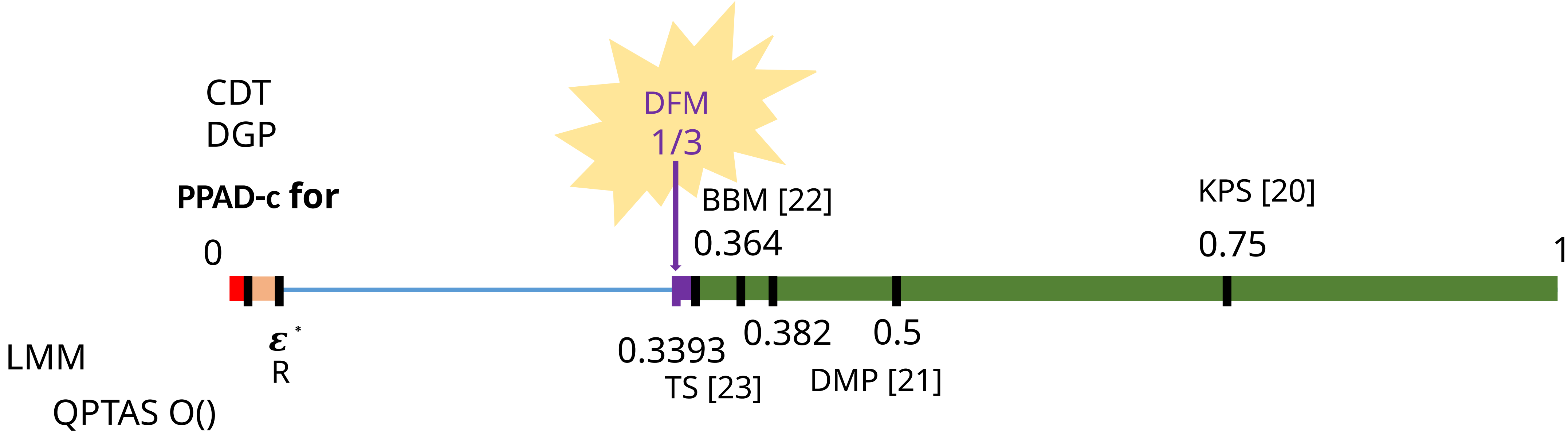
- Take any pair of strategies (x, y)
- Randomly sample $\log n/\epsilon^2$ pure strategies
- Play the sampled strategies uniformly
- The resulting payoffs will be within ϵ of the originals w.h.p.

This is the best we can hope assuming the Exponential Time Hypothesis for PPAD! [32]

[32] Settling the complexity of computing approximate two-player Nash equilibria. Rubinstein

The story so far

Given an n x n bimatrix game (R, C), compute an – NE in polynomial time w.r.t. n



TS algorithm [23] (2007)

- Works quite well in practice
- Maybe better analysis is possible?

TS analysis is proven to be tight!! [24]

[20] Polynomial Algorithms for Approximating Nash Equilibria of Bimatrix Games. Kontogiannis, Panagopoulou, Spirakis
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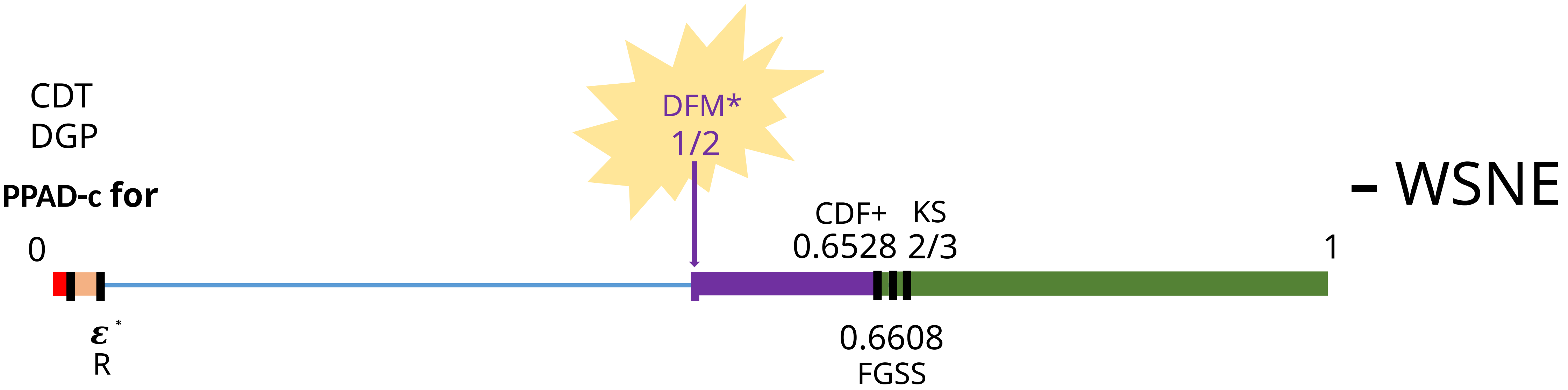
Given an n x n bimatrix game (R, C), compute an – WSNE in polynomial time w.r.t. n



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[FGSS] Approximate Well-Supported Nash Equilibria Below Two-Thirds. Fearnley, Goldberg, Savani, Sorensen

[CDF+] Distributed Methods for Computing Approximate Equilibria. Czumaj, Deligkas, Fasoulakis, Fearnley, Jurdzinski, Savani

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Epilogue

- Nash equilibria form the fundamental solution in games
- Hard to compute any of them!
- Harder to compute an NE with specific properties!
- Many ways to improve the results!
  - New algorithms for approximate NE
  - Better lower bounds
  - Identify tractable cases

Challenging but important (and fun) problems

Thank you!